Equidistant Chebyshev centers

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1 Preliminaries

For ease of reference, we list the results we demonstrate in the paper at the beginning of the document. The main result we show here is the following.

Theorem. *If three distinct $x_1, x_2, x_3 \in \mathbb{F}_2^n$ have equal Hamming weight, then there exists a Chebyshev center $y \in \mathbb{F}_2^n$ that is the same Hamming distance from each x_i.*

The writing of a solution to this problem is in response to a question posed by Yihan Zhang five years ago [2, 3]. This question has relevance to Zhang's research on list decoding [4, 5]. We reproduce necessary definitions to understand the problem statement in this section. Here, $\mathbb{F}_2$ is the finite field with two elements, and the Hamming distance d between two vectors x and y is the number of coordinates i where $x_i \neq y_i$.

Definition 1.1. Given three distinct points $x_1, x_2, x_3 \in \mathbb{F}_2^n$, with Hamming distance d, the *Chebyshev radius* of them is

$$\min_{y \in \mathbb{F}_2^n} \max_{i \in \{1,2,3\}} d(x_i, y).$$

Definition 1.2. The *Chebyshev centers* of x_1, x_2, x_3 is the set of y which achieve the optimal value of the above radius.

In a continuous setting, the Chebyshev center is the center of the largest ball inscribed by the set of points given. Note also that the Chebyshev radius and Chebyshev center may be defined for n points, instead of only three [1, Section 4.3.1]. However, we restrict Definition 1.2 to three points in order to stay consistent with the question as written.

*These results were produced not in affiliation with, and without any resources of, the current place of work of the author.

2 Results

The approach for proving the main theorem uses strong induction. The idea behind our proof is to show that there is always a way to guarantee the existence of coordinates $\{j\}_{j\in\{1,\ldots,n\}}$ which do not affect the equal Hamming weight of x_1, x_2, x_3 when disregarded. We then consider a new triple, formed by restricting to every other coordinate outside of this set. Then, this new triple of shortened vectors will have an equidistant Chebyshev center. From this equidistant Chebyshev center, we may then construct a Chebyshev center which has the length of the original vectors $\{x_i\}$, and is equidistant to them.

This argument is supplemented by Lemma 1. The purpose of Lemma 1 is to show that at least one of the following five configurations must exist within sufficiently long vector triples $\{x_i\}$. Notice that each of these coordinate configurations has equal Hamming weight among $\{x_i\}$. So, informally, the following lemma shows that these are the "atomic" ways to maintain equal weight among three vectors over $\mathbb{F}_2$.

Lemma 1. *The j^{th} coordinate of x_i is written x_i^j. Suppose that three distinct $x_1, x_2, x_3 \in \mathbb{F}_2^n$ have equal Hamming weight and that $n > 6$. If no component is common among them (i.e., for no j are all x_i^j either 1 or 0), then there must exist either a pair j_1, j_2 or triple j_1, j_2, j_3 of coordinates which have one of the following 5 sorts of entries for x_1, x_2, x_3 respectively.*

$$\begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \left\{ \begin{array}{ccc|cc|cc|cc|ccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right.$$

Proof. We wish to show that at least one of the five above configurations must exist; to do this, assume that none of the last four hold, and will show the first configuration must exist.

Assume for the sake of contradiction that exactly two of the x_i have entries of 1 in some coordinate j. Then, there is no column with a 1 in any other coordinates of the third x_i. We rephrase this in an example, without loss of generality:

If the x_i entries of index j appear as $\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}$, then there are no indices of the following:

- $\begin{matrix} x_1 = & (\ldots & 0 & \ldots) \\ x_2 = & (\ldots & 0 & \ldots) \\ x_3 = & (\ldots & 1 & \ldots) \end{matrix}$, since the second configuration does not occur,
- $\begin{matrix} x_1 = & (\ldots & 1 & \ldots) \\ x_2 = & (\ldots & 0 & \ldots) \\ x_3 = & (\ldots & 1 & \ldots) \end{matrix}$ or $\begin{matrix} x_1 = & (\ldots & 0 & \ldots) \\ x_2 = & (\ldots & 1 & \ldots) \\ x_3 = & (\ldots & 1 & \ldots) \end{matrix}$, since the fifth does not occur, and
- $\begin{matrix} x_1 = & (\ldots & 1 & \ldots) \\ x_2 = & (\ldots & 1 & \ldots) \\ x_3 = & (\ldots & 1 & \ldots) \end{matrix}$, since we assumed no common entry in any coordinate.

However, this would mean that the third x_i can never have an entry of 1, a contradiction with the assumption of equal Hamming weight among x_1, x_2, x_3.

Since no index j can have two of x_1^j, x_2^j, x_3^j equal 1, and since by assumption there is no coordinate where all three vectors are 0 or 1, we then know that for each coordinate j there is exactly one x_i with entry 1. We now must guarantee that at some coordinate in each x_i there is a 1; since there is only one vector with all zero coordinates, for each x_i to have the same weight and still be distinct from each other, there must be at least one coordinate j_i where $x_i^{j_i} = 1$, and gathering one such coordinate for each i gives us our first configuration. ■

With this, we proceed with the strong induction argument to prove the main result. Note that since the configurations in Lemma 1 utilize two or three columns, that there are multiple base cases to include in order to fully use the lemma.

Theorem 1. *If three distinct $x_1, x_2, x_3 \in \mathbb{F}_2^n$ have equal Hamming weight, then there exists a Chebyshev center $y \in \mathbb{F}_2^n$ that is equidistant from them; that is,*

$$d(x_1, y) = d(x_2, y) = d(x_3, y).$$

Proof. We proceed by using strong induction, and include multiple base cases.

<u>Base case $n = 3$</u>: The only triples in $\mathbb{F}_2^3$ that have equal weight are the following.

$$\begin{array}{ll} x_1 = (1,0,0) & \quad x_1 = (1,1,0) \\ x_2 = (0,1,0) & \quad x_2 = (1,0,1) \\ x_3 = (0,0,1) & \quad x_3 = (0,1,1) \end{array}$$

Each of these triples have Chebyshev centers $y = (0,0,0)$ and $y = (1,1,1)$ respectively. Each of these two Chebyshev centers are of Hamming distance 1 from each member of their respective triples.

<u>Base cases $n = 4, 5, 6$</u>: Proceed similarly with computation. A Python script to test reasonably small cases is included in Appendix A.

<u>Inductive step</u>: Assume that $n > 6$. Assume that for all $k < n$ that for any triple $x_1, x_2, x_3 \in \mathbb{F}_2^k$ of equal Hamming weight that there exists a Chebyshev center $y \in \mathbb{F}_2^k$ equidistant to the three of them. We will show that for any triple $x_1, x_2, x_3 \in \mathbb{F}_2^n$ that there exists an equidistant Chebyshev center $y \in \mathbb{F}_2^n$.

<u>Case 1</u>: Assume that there exists some index with an entry common to all x_1, x_2, x_3. That is, if we label the j^{th} component of x_i as x_i^j, then for some $1 \leq j \leq n$ we have that either $x_i^j = 1$ for all i, or then $x_i^j = 0$ for all i. As a visual example, this appears as the following when the common entry is a 0.

$$\begin{array}{l} x_1 = (x_1^1, x_1^2, \ldots, x_1^{j-1}, 0, x_1^{j+1}, \ldots, x_1^n) \\ x_2 = (x_2^1, x_2^2, \ldots, x_2^{j-1}, 0, x_2^{j+1}, \ldots, x_2^n) \\ x_3 = (x_3^1, x_3^2, \ldots, x_3^{j-1}, 0, x_3^{j+1}, \ldots, x_3^n) \end{array}$$

Consider then the triple constructed by restricting to all coordinates except j, forming the triple

$$\begin{aligned} x_1' &= (x_1^1, x_1^2, \ldots, x_1^{j-1}, x_1^{j+1}, \ldots, x_1^n) \\ x_2' &= (x_2^1, x_2^2, \ldots, x_2^{j-1}, x_2^{j+1}, \ldots, x_2^n) \\ x_3' &= (x_3^1, x_3^2, \ldots, x_3^{j-1}, x_3^{j+1}, \ldots, x_3^n) \end{aligned}$$

Since this new triple has members of length $n - 1 < n$, we know by the inductive hypothesis that there exists a Chebyshev center $y' \in \mathbb{F}_2^{n-1}$ equidistant to x_1', x_2', x_3'. We can then construct a Chebyshev center from this y, by inserting an appropriate coordinate; if the common component was 0 (resp. 1), insert a 0 (resp. 1) in y' to make $y \in \mathbb{F}_2^n$, which is then an equidistant Chebyshev center of x_1, x_2, x_3.

Case 2: Assume there exists no index with common entry among x_1, x_2, x_3. Then, for each index j, exactly one or two of x_1^j, x_2^j, x_3^j are 1. Just as in the previous case, we will consider methods of restricting the triple $\{x_i\}$ to an equidistant triple $\{x_i'\}$ of size $k < n$. In order to do this, we again consider ways of removing sets of indices.

We showed in Lemma 1 that there must occur a pair or triple of coordinates that exhibit one of five configurations. Restricting to coordinates other than that pair (or triple), we have vectors $x_1', x_2', x_3' \in \mathbb{F}_2^{n-2}$ (or $\mathbb{F}_2^{n-3}$) which are equal Hamming weight, and so have an equidistant Chebyshev center $y' \in \mathbb{F}_2^{n-2}$ (or $\mathbb{F}_2^{n-3}$) by the inductive hypothesis. This is true even though we are cutting out two or three columns, given base cases through $n = 6$.

Then returning to x_1, x_2, x_3, construct y with the entries of y' in the coordinates we previously restricted to, and a 1 in the coordinates which were removed in the restriction. Then, y is an equidistant Chebyshev center for the triple x_1, x_2, x_3, concluding the second of our two cases. ∎

References

[1] Stephen P Boyd and Lieven Vandenberghe. *Convex optimization.* Cambridge university press, 2004.

[2] Yihan Zhang. Does there always exist a Chebyshev center of three constant weight points in $\mathbb{F}_2^n$ which is equidistant? https://math.stackexchange.com/questions/3159349/does-there-always-exist-a-chebyshev-center-of-three-constant-weight-points-in. Accessed: 2024-03-21.

[3] Yihan Zhang. Yihan Zhang. https://sites.google.com/view/yihan/?pli=1. Accessed: 2024-03-21.

[4] Yihan Zhang, Amitalok J Budkuley, and Sidharth Jaggi. Generalized list decoding. In *2020 Information Theory and Applications Workshop (ITA)*, pages 51–1. IEEE, 2020.

[5] Yihan Zhang and Shashank Vatedka. Multiple packing: Lower bounds via error exponents. *IEEE Transactions on Information Theory*, 2023.

A Base cases in Python

```
# Equidistant Chebyshev center
# Author: Aidan W. Murphy

# Purpose: Determine if all triples of equal-Hamming-weight vectors in
#          (F_2)^n have a Chebyshev center which is equidistant from them.
# Inputs: n - length of vectors
# Outputs: True/False - truth of all vector triples in (F_2)^n having such
#          a Chebyshev center

# Importing packages and functions
import numpy as np;
from itertools import combinations;

# Defining functions
def binaryHammingDistance(x,y):
    # Inputs:
    #    x,y - binary lists of the same length
    # Output:
    #    S - the number of differing coordinate entries
    S = [];
    for i in range(len(x)):
        S.append((x[i] + y[i]) % 2);
    return(sum(S));

def allBinaryStrings(n,arr,i,blankList):
    # Inputs:
    #    n - length of strings to generate
    #    i - iterate, for recursion
    # Output:
    #    blankList - list of all strings
    s = [];
    if i == n:
        for i in range(0,n):
            s.append(arr[i]);
        blankList.append(s);
        s = [];
        return

    arr[i] = 0;
    allBinaryStrings(n,arr,i+1,blankList);
    arr[i] = 1;
    allBinaryStrings(n,arr,i+1,blankList);
    return(blankList);

def equalWeightTriples(stringList):
    # Inputs:
    #    stringList - a list of binary lists of equal length
    # Output:
    #     tripleList - all possible equal weight triples of the original
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    #                    list
    tripleList = [];
    combos = combinations(stringList,3);
    combos = list(combos);

    for i in range(len(combos)):
        x1 = combos[i][0];
        x2 = combos[i][1];
        x3 = combos[i][2];
        if (sum(x1) == sum(x2) and sum(x2) == sum(x3)):
            tripleList.append([x1,x2,x3]);
    return(tripleList);

def chebyshevCenters(stringList):
    # Inputs:
    #     stringList - a list of binary lists of equal length
    # Output:
    #     centers - list of all Chebyshev centers of the input collection
    distanceList = [];
    for i in range(2**n):
        tempList = [];
        for j in range(len(stringList)):
            d = binaryHammingDistance(S[i],stringList[j]);
            tempList.append(d);
        distanceList.append(tempList);

    for i in range(len(distanceList)):
        distanceList[i] = max(distanceList[i]);

    m = min(distanceList);
    centers = [];
    for i in range(len(distanceList)):
        if distanceList[i] == m:
            centers.append(S[i]);
    return(centers);

# Main program
n = 4; # set the length of vectors to search over

S = allBinaryStrings(n,[None] * n,0,[]); # generate all binary strings
E = equalWeightTriples(S); # identify all equal weight triples

TFList = []; # list to hold results of search
for trip in E: # for all equal weight triples...
    CC = chebyshevCenters(trip); # find Chebyshev centers
    equidistantCenters = [];
    for cent in CC: # and for those Chebyshev centers...
        if ((binaryHammingDistance(cent,trip[0]) == \
            binaryHammingDistance(cent,trip[1])) and \
            (binaryHammingDistance(cent,trip[1]) == \
            binaryHammingDistance(cent,trip[2]))): # determine which are
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            #                                         equidistant,
            equidistantCenters.append(cent); # and record it, if the
            #                                   center is equidistant.

    # uncomment to see triples and their equidistant centers
    # print("Triple: ",trip,"   |    Equidistant Centers: ",
                                             equidistantCenters)

    if equidistantCenters == []: # If there are no equidistant centers...
        TFList.append(False); # "there are equidistant centers" is False.
    else: # Otherwise...
        TFList.append(True); # "there are equidistant centers" is True.

# return False if any of the above are False, and True otherwise.
isThereACenterforallTriples = bool(np.prod(TFList));
print(isThereACenterforallTriples)
```